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Spin Stability of Undamped Flexible Structures Rotating About the Minor Axis

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A method is presented for determining the nonlinear stability of undamped flexible structures spinning about the axis of minimum moment of inertia. Equations of motion are developed for structures that are free of applied forces and moments. The development makes use of a floating reference frame which follows the overall rigid body motion. Within this frame, elastic deformations are assumed to be given functions of n generalized coordinates. A transformation of variables is devised which shows the equivalence of the equations of motion to a Hamiltonian system with $n+1$ degrees of freedom. Using this equivalence, stability criteria are developed based on the normal form of the Hamiltonian. It is shown that a motion which is spin stable in the linear approximation may be unstable when nonlinear terms are included. A stability analysis of a simple flexible structure is provided to demonstrate the application of the stability criteria. Results from numerical integration of the equations of motion are shown to be consistent with the predictions of the stability analysis.

Introduction

A CLASSICAL result of dynamics states that the rotational motion of a rigid body is stable for spin about either the axis of minimum (minor) or maximum (major) moment of inertia when applied moments are absent. Such a general statement cannot be made for flexible bodies. Here, stability depends on a number of additional factors including the spin rate, geometry, and stiffness of the body.

A great deal of interest in the spin stability of flexible structures was motivated by the unexpected performance of the Explorer I satellite. Initially spun up about its minor axis, radio signals later indicated that the satellite was in a tumbling motion after only one complete orbit. Bracewell and Garriott[1] are attributed with providing a simple physical explanation for the observed motion based on energy considerations.

A solid mathematical basis for the stability analysis of rotating bodies was provided by the work of Pringle.[2] Using the direct method of Lyapunov, basic theorems were established on the stability of damped mechanical systems with connected moving parts. Among these was the so-called maximum axis rule which states that for completely damped systems a motion of simple spin can only be stable about the major axis. Other examples of the application of Lyapunov's direct method include, among others, Hughes and Fung,[3] Meirovitch,[4] and Teixeira-Filho and Kane.[5] More recently, Krishnaprasad and Marsden[6] and Simo et al.[7] applied related techniques to the stability analysis of continuous beam models.

Use of Lyapunov's direct method for proving spin stability about the major axis typically involves construction of a Lyapunov function from the angular momentum integrals (constants of motion) and the total energy. Such an approach cannot be used to show stability for spin about the minor axis, even for undamped structures. At least one more integral of motion is required in order to construct a Lyapunov function. It turns out, however, that integrals in addition to those of energy and momentum are exceptional.

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To the best of the authors' knowledge, the nonlinear (Lyapunov) stability of rotating flexible structures has only been proven for cases of spin about the major axis. The contribution of the present work is toward the development of a means to assess the nonlinear stability of undamped flexible structures spinning about the minor axis. Results are applicable to discrete models of elastic bodies that are free from applied forces and moments. Dual spin satellites and multibody configurations are not considered. This work was originally motivated by a need to predict the attitude stability of a lightweight re-entry vehicle decoy.

Equations of motion derived in the first section from fundamental principles of dynamics are subsequently reformulated as a Hamiltonian system using a transformation of variables. Stability criteria are developed specifically for motions that are stable in the linear approximation. A stability analysis is made of the motion of a simple flexible structure spinning about its minor axis. Simulations of the equations of motion are presented as a confirmation of the stability analysis.

Equations of Motion

A system of interconnected particles each of mass m^i ($i=1, \ldots, N$) is depicted in Fig. 1. Also shown in the figure are a floating reference frame B and an inertial frame A. Orthogonal, dextral sets of unit vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ and $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ are fixed in B and A, respectively. The angular velocity vector of B in A is denoted by $\boldsymbol{\omega}$. The position vector from the origin O of B to the ith particle when the system is undeformed is denoted by $\boldsymbol{r}^i$. The displacement vector $\boldsymbol{u}^i$ of the ith particle from its undeformed position is assumed to be a function of dimensionless generalized coordinates $q_1, \ldots, q_n$.

The origin of B is chosen as the mass center of the system. Consequently, O remains fixed in an inertial frame since the system is assumed to be free of applied forces. The orientation of B in A depends on the particular choice of the floating frame.[8] Regardless of this choice, the intention is for B to follow the nominal rigid body motion of the system. In addition, the unit vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ are assumed to be parallel to the central principal axes of the system for motions of simple spin about a principal axis.

Three notational conventions are adopted herein for convenience. The first is the placement of numbers above the equality sign in an equation. For example, the notation $y \overset{(1,2)}{=} mx + b$

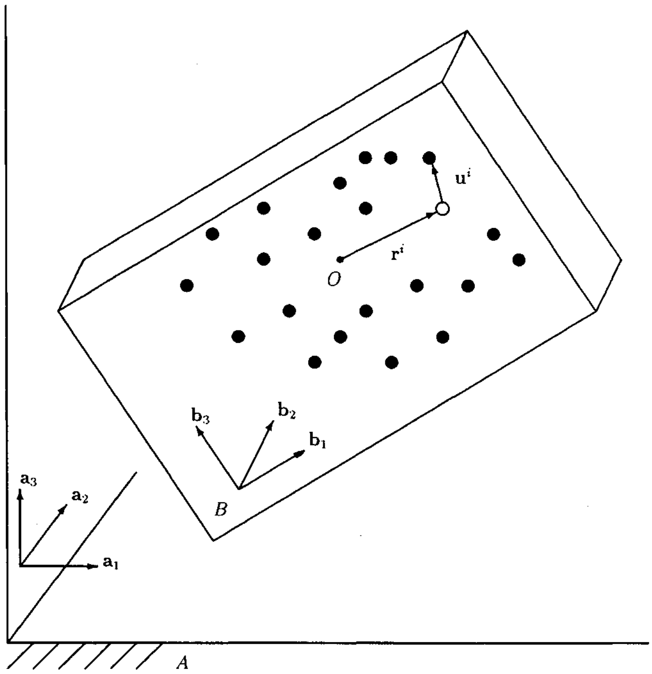

Fig. 1 Sketch of the system of particles and reference frames.

indicates that the result is obtained with reference to Eqs. (1) and (2). The second convention concerns the notation used for the measure numbers of a vector. For any vector v, one has $v_k \equiv v \cdot b_k$ for $k = 1, 2, 3$. Finally, Einstein's summation convention is employed where convenient. With this convention, the repeated appearance of an index implies summation over all of the possible values of the index.

To begin, the system kinetic energy T is defined as

$$T = \frac{1}{2} \sum_{i=1}^{N} m^i v^i \cdot v^i \tag{1}$$

where v^i denotes the velocity of the ith particle in A. Using a basic kinematical relationship, one obtains

$$v^i = \left[\frac{\partial u_k^i}{\partial q_j} \dot{q}_j + \epsilon_{klm} \omega_l (r_m^i + u_m^i) \right] b_k \tag{2}$$

where ϵ_{klm} is the permutation symbol and $(\dot{\ })$ denotes time differentiation.

Under the assumption that the system is free of applied moments, the Lagrangian equations for quasicoordinates associated with ω_1, ω_2, ω_3 are[9]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right) = \omega_3 \frac{\partial T}{\partial \omega_2} - \omega_2 \frac{\partial T}{\partial \omega_3} \tag{3}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_2}\right) = \omega_1 \frac{\partial T}{\partial \omega_3} - \omega_3 \frac{\partial T}{\partial \omega_1} \tag{4}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_3}\right) = \omega_2 \frac{\partial T}{\partial \omega_1} - \omega_1 \frac{\partial T}{\partial \omega_2} \tag{5}$$

Equations (3-5) are analogous to Euler's equations for a rigid body and can be derived alternatively from the angular momentum principle. Indeed, if the angular momentum vector of the system about O is denoted by h, one can show from Eqs. (1) and (2) that

$$h_k = \frac{\partial T}{\partial \omega_k} \qquad (k = 1, 2, 3) \tag{6}$$

Furthermore, it follows directly from Eqs. (3-6) that the magnitude of the angular momentum vector h defined as

$$h = \sqrt{h_1^2 + h_2^2 + h_3^2} \tag{7}$$

is constant. Equation (7) is used later to reduce the number of equations of motion by one.

The remaining equations of motion for the undamped system are given by the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{\partial T}{\partial q_j} - \frac{\partial U}{\partial q_j} \qquad (j = 1, \ldots, n) \tag{8}$$

where U denotes the strain energy of the system. The strain energy is a function solely of $q_1, \ldots, q_n$.

Additional kinematic variables must be introduced if the orientation of B in A is required. Several possibilities exist including Euler angles, direction cosines, Euler parameters, and Rodrigues parameters. For the purposes of this paper, the introduction of such variables is not necessary.

It is clear from examination of Eqs. (1) and (2) that the kinetic energy can be expressed equivalently in matrix notation as

$$T = ½[\omega_1\ \omega_2\ \omega_3\ \dot{q}_1 \ldots \dot{q}_n] M [\omega_1\ \omega_2\ \omega_3\ \dot{q}_1 \ldots \dot{q}_n]^T \tag{9}$$

where the matrix M is symmetric, positive definite, and a function of $q_1, \ldots, q_n$.

Upon differentiation of Eq. (9) with respect to ω_1, ω_2, ω_3 and $\dot{q}_1, \ldots, \dot{q}_n$ and using Eq. (6), one obtains

$$[h_1\ h_2\ h_3\ p_1 \ldots p_n]^T = M[\omega_1\ \omega_2\ \omega_3\ \dot{q}_1 \ldots \dot{q}_n]^T \tag{10}$$

where momenta variables $p_1, \ldots, p_n$ are defined as

$$p_j = \frac{\partial T}{\partial \dot{q}_j} \qquad (j = 1, \ldots, n) \tag{11}$$

Premultiplication of Eq. (10) by M^{-1} yields

$$[\omega_1\ \omega_2\ \omega_3\ \dot{q}_1 \ldots \dot{q}_n]^T = M^{-1}[h_1\ h_2\ h_3\ p_1 \ldots p_n]^T \tag{12}$$

or, equivalently,

$$\omega_k = m_{kl}^{-1} h_l + m_{k,s+3}^{-1} p_s \qquad (k = 1, 2, 3) \tag{13}$$

$$\dot{q}_j = m_{j+3,l}^{-1} h_l + m_{j+3,s+3}^{-1} p_s \qquad (j = 1, \ldots, n) \tag{14}$$

It is desirable to express the equations of motion entirely in terms of the generalized coordinates and momenta variables. To this end, let

$$\hat{T} = ½[h_1\ h_2\ h_3\ p_1 \ldots p_n] M^{-1} [h_1\ h_2\ h_3\ p_1 \ldots p_n]^T \tag{15}$$

Differentiation of Eq. (15) with respect to q_j and p_j yields

$$\frac{\partial \hat{T}}{\partial q_j} = \frac{1}{2}[h_1\ h_2\ h_3\ p_1 \ldots p_n] \frac{\partial M^{-1}}{\partial q_j} [h_1\ h_2\ h_3\ p_1 \ldots p_n]^T$$

$$\overset{(12)}{=} -\frac{1}{2}[\omega_1\ \omega_2\ \omega_3\ \dot{q}_1 \ldots \dot{q}_n] \frac{\partial M}{\partial q_j} [\omega_1\ \omega_2\ \omega_3\ \dot{q}_1 \ldots \dot{q}_n]^T$$

$$\overset{(8,9,11)}{=} -\frac{d}{dt} p_j - \frac{\partial U}{\partial q_j} \qquad (j = 1, \ldots, n) \tag{16}$$

and

$$\frac{\partial \hat{T}}{\partial p_j} \overset{(14)}{=} \frac{d}{dt} q_j \qquad (j = 1, \ldots, n) \tag{17}$$

Equations (16) and (17) can be expressed alternatively as

$$\frac{d}{dt} q_j = \frac{\partial E}{\partial p_j} \qquad (j = 1, \ldots, n) \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_j = -\frac{\partial E}{\partial q_j} \qquad (j = 1, \ldots, n) \tag{19}$$

where

$$E = \hat{T} + U \tag{20}$$

The motion whose stability is of interest is one of simple spin in which $q_1, \ldots, q_n$ remain equal to zero and the angular momentum vector and $\boldsymbol{b}_1$ are in the same direction. As the case may be, it is useful to solve Eq. (7) for h_1 to obtain

$$h_1 = \sqrt{h^2 - h_2^2 - h_3^2} \tag{21}$$

Use of the positive square root in Eq. (21) is valid as long as $h_2^2 + h_3^2$ remains less than h^2. This requirement is satisfied for motions which remain close to a simple spin about the first principal axis.

Substitution of Eqs. (15) and (21) into Eq. (20) and subsequent differentiation with respect to h_2 yields

$$\begin{aligned}\frac{\partial E}{\partial h_2} &= \frac{1}{h_1}\left[- h_2(m_{1l}^{-1}h_l + m_{1,m+3}^{-1}p_m) + h_1(m_{2l}^{-1}h_l + m_{2,m+3}^{-1}p_m)\right.\\ &\overset{(13)}{=} \frac{(\omega_2 h_1 - \omega_1 h_2)}{h_1}\\ &\overset{(5,6)}{=} \frac{1}{h_1}\frac{\mathrm{d}}{\mathrm{d}t} h_3\end{aligned} \tag{22}$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t} h_3 = \sqrt{h^2 - h_2^2 - h_3^2}\,\frac{\partial E}{\partial h_2} \tag{23}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t} h_2 = -\sqrt{h^2 - h_2^2 - h_3^2}\,\frac{\partial E}{\partial h_3} \tag{24}$$

At this point in the development, it is convenient to express the equations of motion in a dimensionless form. To this end, we define the spin rate Ω as

$$\Omega = h/I \tag{25}$$

where I denotes the (1,1) element of the matrix M for $q_1, \ldots, q_n$ all equal to zero. In other words, I is simply the moment of inertia about the first principal axis of the undeformed system.

Defining $\tau = \Omega t$, $G = E/(h\Omega)$, $m_k = h_k/h$, $x_j = q_j$, and $y_j = p_j/h$, the equations of motion assume the dimensionless form

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x_j \overset{(18)}{=} \frac{\partial G}{\partial y_j} \qquad (j = 1, \ldots, n) \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} y_j \overset{(19)}{=} -\frac{\partial G}{\partial x_j} \qquad (j = 1, \ldots, n) \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} m_3 \overset{(23)}{=} \sqrt{1 - m_2^2 - m_3^2}\,\frac{\partial G}{\partial m_2} \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} m_2 \overset{(24)}{=} -\sqrt{1 - m_2^2 - m_3^2}\,\frac{\partial G}{\partial m_3} \tag{29}$$

Notice that Eqs. (26-29) would form a Hamiltonian system with $n + 1$ degrees of freedom were it not for the presence of $\sqrt{1 - m_2^2 - m_3^2}$ in Eqs. (28) and (29). Transformation of these equations to an equivalent Hamiltonian system is the topic of the following section.

Transformation to Canonical Form

New variables x_0 and y_0 are introduced as functions of m_2 and m_3. Letting H denote the function G expressed in terms of the new variables, one has

$$H(x_0, \ldots, x_n, y_0, \ldots, y_n) = G(m_2, m_3, x_1, \ldots, x_n, y_1, \ldots, y_n) \tag{30}$$

Application of the chain rule for differentiation to Eq. (30) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x_j \overset{(26)}{=} \frac{\partial H}{\partial y_j} \qquad (j = 1, \ldots, n) \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} y_j \overset{(27)}{=} -\frac{\partial H}{\partial x_j} \qquad (j = 1, \ldots, n) \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x_0 \overset{(28-29)}{=} \sqrt{1 - m_2^2 - m_3^2}\left[\frac{\partial x_0}{\partial m_3}\frac{\partial y_0}{\partial m_2} - \frac{\partial x_0}{\partial m_2}\frac{\partial y_0}{\partial m_3}\right]\frac{\partial H}{\partial y_0} \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} y_0 \overset{(28-29)}{=} -\sqrt{1 - m_2^2 - m_3^2}\left[\frac{\partial x_0}{\partial m_3}\frac{\partial y_0}{\partial m_2} - \frac{\partial x_0}{\partial m_2}\frac{\partial y_0}{\partial m_3}\right]\frac{\partial H}{\partial x_0} \tag{34}$$

Equations (31-34) are equivalent to the Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x_j = \frac{\partial H}{\partial y_j} \qquad (j = 0, \ldots, n) \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} y_j = -\frac{\partial H}{\partial x_j} \qquad (j = 0, \ldots, n) \tag{36}$$

provided that

$$\frac{\partial x_0}{\partial m_3}\frac{\partial y_0}{\partial m_2} - \frac{\partial x_0}{\partial m_2}\frac{\partial y_0}{\partial m_3} = \frac{1}{\sqrt{1 - m_2^2 - m_3^2}} \tag{37}$$

To obtain a solution to the partial differential equation given by Eq. (37), consider the transformation of variables

$$x_0 = m_3 c(m_2^2 + m_3^2) \tag{38}$$

$$y_0 = m_2 c(m_2^2 + m_3^2) \tag{39}$$

and its inverse

$$m_2 = y_0 d(x_0^2 + y_0^2) \tag{40}$$

$$m_3 = x_0 d(x_0^2 + y_0^2) \tag{41}$$

where c and d are functions of the specified arguments. Substitution of Eqs. (38) and (39) into Eq. (37) yields the nonlinear ordinary differential equation

$$2sc(s)c'(s) + [c(s)]^2 = 1/\sqrt{1 - s} \tag{42}$$

where

$$s = m_2^2 + m_3^2 \tag{43}$$

A solution to Eq. (42), obtained using the software package Mathematica,[10] is

$$c(s) = \sqrt{\frac{2(1 - \sqrt{1 - s})}{s}} \tag{44}$$

It can be verified that Eq. (44) is the solution to Eq. (42) for the initial condition $c(0) = 1$. Thus, in the linear approximation, $x_0 = m_3$ and $y_0 = m_2$.

Substitution of Eqs. (38) and (39) and (43) and (44) into Eqs. (40) and (41) yields

$$d(x_0^2 + y_0^2) = \sqrt{1 - \frac{x_0^2 + y_0^2}{4}} \tag{45}$$

The procedure for transforming the original equations of motion to an equivalent Hamiltonian system is summarized as follows. It is assumed that expressions for the kinetic and strain energies of the system are available from a mathematical model of the structure.

1) Construct the matrix M using the expression for the kinetic energy along with Eq. (9).

2) Construct the function G, given by

$$G = \tfrac{1}{2}\{[\sqrt{1 - m_2^2 - m_3^2}\; m_2\; m_3\; y_1 \ldots y_n](M/I)^{-1} \times [\sqrt{1 - m_2^2 - m_3^2}\; m_2\; m_3\; y_1 \ldots y_n]^T\} + U/(I\Omega^2) \qquad (46)$$

3) Construct the Hamiltonian H by substituting Eqs. (40) and (41) for m_2 and m_3 into Eq. (46). This procedure is applied to an example problem later in the paper.

The net result of the transformation is to permit the expression of the original equations of motion as a Hamiltonian system with $n + 1$ degrees of freedom. Such a result for a rigid body ($n = 0$) has been attributed to Andoyer (see Ref. 11). The present transformation differs fundamentally from Andoyer's by never requiring the introduction of Euler angles.

Stability Analysis

Recall that the motion of interest is one of simple spin about an axis parallel to the unit vector $\boldsymbol{b}_1$. Accordingly, one possible measure of the departure from simple spin is the angle ϕ between $\boldsymbol{b}_1$ and the angular momentum vector. Using a property of the vector cross product, one obtains

$$\begin{aligned} |\sin\phi| &= \|(\boldsymbol{b}_1 \times \boldsymbol{h})\|/\|\boldsymbol{h}\| \\ &= \sqrt{m_2^2 + m_3^2} \\ &\overset{(40,41)}{=} \sqrt{x_0^2 + y_0^2}\; d(x_0^2 + y_0^2) \end{aligned} \qquad (47)$$

It is evident from Eq. (47) that small magnitudes of x_0 and y_0 imply small departures from simple spin as measured by the angle ϕ. Likewise, small magnitudes of $x_1, \ldots, x_n$ imply small elastic deformations. Thus, one reasonable definition of stability is that the null solution

$$x_j(\tau) = 0 \qquad 0 \le \tau < \infty, \qquad j = 0, \ldots, n \qquad (48)$$

$$y_j(\tau) = 0 \qquad 0 \le \tau < \infty, \qquad j = 0, \ldots, n \qquad (49)$$

to Eqs. (35) and (36) is stable in the sense of Lyapunov. A simple spin may be accompanied by constant elastic deformations caused by centrifugal loading. Nevertheless, $x_1, \ldots, x_n$ can be defined in such a manner that they are all zero for the simple spin of interest.

The equations of motion for a rotating flexible structure given by Eqs. (35) and (36) are inherently nonlinear. Nonlinearities arise because of the presence of cubic and higher order terms in the expansion of the Hamiltonian. Such is the case even for the simple problem of rigid body motion.

Determination of stability for nonlinear Hamiltonian systems may or may not be a simple matter. In many instances, the stability of an equilibrium can be determined from a linear analysis of the nonlinear differential equations.

If the general solution to the linear equations involves a term with exponential growth, then the equilibrium is unstable. This result holds for the nonlinear differential equations as well. If the general solution to the linear equations does not contain any terms with exponential growth, then no conclusions can be made regarding stability for the nonlinear system. That is, an equilibrium stable in the linear approximation may not be stable when nonlinear terms in Hamilton's equations are included.

Before proceeding further, certain aspects of Hamiltonian systems pertinent to later discussions are reviewed. Assuming that $(x, y) = (0, 0)$ is an equilibrium point and H can be expanded as a Taylor series, one has

$$H(x, y) = H_0 + H_2(x, y) + H_3(x, y) + \cdots \qquad (50)$$

where $H_s(x, y)$ is a homogeneous polynomial of degree s in the canonical variables $(x, y) = (x_0, \ldots, x_n, y_0, \ldots, y_n)$. It is possible through the use of canonical transformations to put a Hamiltonian into its so-called normal form.[11,12] Assuming stability in the linear approximation, a Hamiltonian normalized to degree k satisfies the properties

$$H_2(x, y) = \sum_{j=0}^{n} \frac{\Omega_j}{2}(x_j^2 + y_j^2) \qquad (51)$$

and

$$[H_j, H_2]_{(x,y)} = 0 \qquad (j = 3, 4, \ldots, k) \qquad (52)$$

where [,] denotes the Poisson bracket of the indicated functions. The characteristic frequencies $\Omega_0, \ldots, \Omega_n$ appearing in Eq. (51) are said to satisfy a resonance relation of order $l > 0$ if there exist integers $k_0, \ldots, k_n$ such that

$$k_0\Omega_0 + \cdots + k_n\Omega_n = 0 \qquad (53)$$

$$|k_0| + \cdots + |k_n| = l \qquad (54)$$

Consider now a Hamiltonian system with r independent resonance relations of the form

$$K\Omega = 0 \qquad (55)$$

where

$$K = \begin{bmatrix} k_{10} & k_{11} & \cdots & k_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{r0} & k_{r1} & \cdots & k_{rn} \end{bmatrix} \qquad (56)$$

$$\Omega = [\Omega_0 \quad \Omega_1 \quad \cdots \quad \Omega_n]^T \qquad (57)$$

According to the theory of the normal form, formal integrals of the normalized Hamiltonian are given by

$$I_l(x, y) = \frac{1}{2}\sum_{j=0}^{n} c_{jl}(x_j^2 + y_j^2) \qquad (l = 1, \ldots, n + 1 - r) \qquad (58)$$

where the columns of the matrix C span the null space of K. That is,

$$\sum_{j=0}^{n} k_{sj}c_{jl} = 0 \quad \text{for} \quad s = 1, \ldots, r$$

and

$$l = 1, \ldots, n + 1 - r \qquad (59)$$

For the remainder of this discussion, it is assumed that the null solution to Eqs. (35) and (36) is stable in the linear approximation and that the Hamiltonian is normalized to at least degree four. Accordingly, one has

$$H(x, y) = \sum_{j=0}^{n} \frac{\Omega_j}{2}(x_j^2 + y_j^2) + H_3(x, y) + H_4(x, y) + \cdots \qquad (60)$$

The Hamiltonian itself is a Lyapunov function when all of the characteristic frequencies in Eq. (60) are of the same sign. This result follows from the sign definiteness of the quadratic part of the Hamiltonian and the constancy of $H(x, y)$ along trajectories of Eqs. (35) and (36).

Although there are exceptions, the characteristic frequencies are often all positive for spin about the major axis. Consequently, $H(x, y)$ is a Lyapunov function and stability follows directly. It is readily apparent that cubic and higher degree terms in the Hamiltonian have no effect on Lyapunov stability in these cases. An equivalent statement is that stability is determined solely from consideration of the linearized equations.

The situation is considerably different for spin about the minor axis. In such cases, the characteristic frequencies can never all be of the same sign. Consequently, the Hamiltonian is no longer a Lyapunov function and cannot be used to prove stability. It still may be possible to prove stability using Lyapunov's direct method if an integral of motion in addition to the Hamiltonian exists. The existence of additional integrals, however, is exceptional.

The quest for a general method of proving the stability of Hamiltonian systems remains open. Indeed, even relatively simple systems such as the three body problem have defied complete analysis. Elements of two theories having application to the stability analysis of spin about the minor axis are described next.

A significant advancement in the understanding of Hamiltonian systems was provided by the development of KAM theory.[11,13] Named in recognition of the collective work of Kolmogorov, Arnold, and Moser, KAM theory has important applications to the study of stability.

Most Hamiltonian systems are nonintegrable. That is, the number of degrees of freedom exceeds the number of integrals (conserved quantities). Notable exceptions include linear systems and systems with only a single degree of freedom.

Under certain conditions, KAM theory can be used to show that the trajectories of nonintegrable systems often behave as if the systems were integrable. Near a stable equilibrium (in the linear approximation), a large fraction of the phase space is filled by so-called KAM surfaces. Trajectories lying on these surfaces are associated with the regular motion of an integrable system. Such trajectories are stable, remaining within a given neighborhood of the equilibrium for all times.

The fact that the phase space is not entirely filled by KAM surfaces precludes the possibility of inferring Lyapunov stability from KAM theory. An exception is for isoenergetically nondegenerate two-degree-of-freedom systems. Here, trajectories not lying on a KAM surface are constrained forever between adjacent surfaces. For systems with more than two degrees of freedom, a slow drift from the equilibrium known as Arnold diffusion is possible. Nevertheless, from the viewpoint of measure theory, stability can be shown for most initial conditions. This type of stability is commonly referred to as metric stability.[11]

If there are no resonance relations of order four or less, then Eq. (60) can be expressed as

$$H(\rho, \theta) = \sum_{j=0}^{n} \Omega_j \rho_j + \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \beta_{ij} \rho_i \rho_j + \mathcal{O}(|\rho|^{5/2}) \qquad (61)$$

where $\beta_{ij} = \beta_{ji}$, $|\rho| = \rho_0 + \cdots + \rho_n$, and the action-angle variables (ρ, θ) are related to the original variables (x, y) by the canonical transformation

$$x_j = \sqrt{2\rho_j} \sin \theta_j \qquad j = 0, \ldots, n \qquad (62)$$

$$y_j = \sqrt{2\rho_j} \cos \theta_j \qquad j = 0, \ldots, n \qquad (63)$$

A system is said to be nondegenerate if

$$\det \begin{bmatrix} \beta_{00} & \beta_{01} & \cdots & \beta_{0n} \\ \beta_{01} & \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{0n} & \beta_{1n} & \cdots & \beta_{nn} \end{bmatrix} \neq 0 \qquad (64)$$

and isoenergetically nondegenerate if

$$\det \begin{bmatrix} \beta_{00} & \beta_{01} & \cdots & \beta_{0n} & \Omega_0 \\ \beta_{01} & \beta_{11} & \cdots & \beta_{1n} & \Omega_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{0n} & \beta_{1n} & \cdots & \beta_{nn} & \Omega_n \\ \Omega_0 & \Omega_1 & \cdots & \Omega_n & 0 \end{bmatrix} \neq 0 \qquad (65)$$

KAM theory can be applied if a system is either nondegenerate or isoenergetically nondegenerate.

Another approach to stability analysis involves the formal integrals introduced previously. If the formal integrals given by Eq. (58) can be combined with the Hamiltonian to form a positive definite function, then the equilibrium $(x, y) = (0, 0)$ is said to be formally stable. According to Bruno,[14] no example of Lyapunov instability has been found for formally stable systems.

Results

In this section, a method is presented for determining the spin stability of undamped flexible structures rotating about the minor axis. Although the method is applicable to a broad class of problems, there is a small subset of problems to which it is not. This subset consists of systems that are stable in the linear approximation but not formally stable, and have either 1) two or more resonance relations of which one or more are order less than five or 2) no resonance relations of order less than five and satisfying neither Eq. (64) nor Eq. (65). Investigation of stability for these exceptional cases requires a more detailed analysis than is presented here. Such an investigation will be the topic of a future paper.

The method for determining stability is summarized as follows in a step-by-step procedure.

1) Form the Hamiltonian of the system using the procedure given in the section Transformation to Canonical Form.

2) Perform a linear stability analysis of the null solution to Hamilton's equations. Linear stability is a necessary condition for nonlinear stability.

3) Calculate the characteristic frequencies and normal form of the Hamiltonian using existing software.[12] If the integrals given by Eq. (58) can be combined with the Hamiltonian to form a positive definite function, then the spin is formally stable.

4) Determine if any resonance relations of order less than five exist among the characteristic frequencies. If so, proceed to step 5. If not, and either Eq. (64) or Eq. (65) is satisfied, then KAM theory implies metric stability.

5) Apply the stability criteria presented in the following paragraphs to systems which possess a single resonance relation of order two, three, or four.

The null solutions of all systems satisfying the stability criteria presented here are formally stable. The null solutions of two-degree-of-freedom systems ($n = 1$) satisfying these criteria are also Lyapunov stable.

One can assume, without loss of generality, that the integer k_0 in Eq. (53) is positive. Furthermore, it is necessary only to consider cases for which $k_1, \ldots, k_n$ are all greater than or equal to zero. Otherwise, one can easily show that the system is formally stable.

The development for a resonance relation of order two is a simple extension of Sokolskii.[15] In this case, one can consider k_0 and k_1 equal to unity and $k_2, \ldots, k_n$ equal to zero. The normal form of the Hamiltonian is given by

$$H(\rho, \theta) = \sum_{j=0}^{n} \Omega_j \rho_j + \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} \rho_i \rho_j$$

$$+ A \rho_0 \rho_1 \sin 2(\theta_0 + \theta_1 + \psi_1) + B \rho_0 \sqrt{\rho_0 \rho_1} \sin (\theta_0 + \theta_1 + \psi_2)$$

$$+ C \rho_1 \sqrt{\rho_0 \rho_1} \sin (\theta_0 + \theta_1 + \psi_3) + \mathcal{O}(|\rho|^{5/2}) \qquad (66)$$

Let

$$F(\phi) = a + b \sin 2\phi + c \sin \phi + d \cos \phi \tag{67}$$

where

$$a = a_{00} + a_{01} + a_{10} + a_{11} \tag{68}$$

$$b = A \tag{69}$$

$$c = B \cos(\psi_2 - \psi_1) + C \cos(\psi_3 - \psi_1) \tag{70}$$

$$d = B \sin(\psi_2 - \psi_1) + C \sin(\psi_3 - \psi_1) \tag{71}$$

If $F(\phi)$ is not equal to zero for all $0 \le \phi < 2\pi$, then the equilibrium is stable. If $F(\phi^*) = 0$ for some ϕ^* and $F'(\phi^*) \neq 0$, then the equilibrium is unstable.

The following results for resonance relations of third and fourth orders are adopted from Khazin.[16] The normal form of the Hamiltonian is given by

$$H(\rho, \theta) = \sum_{j=0}^{n} \Omega_j \rho_j + \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij}\rho_i\rho_j + A\sqrt{\rho_0^{k_0} \cdots \rho_n^{k_n}} \cos(k_0\theta_0 + \cdots + k_n\theta_n) + \mathcal{O}(|\rho|^{5/2}) \tag{72}$$

The stability criteria for a third-order resonance relation are

$$A = 0 \quad \text{and} \quad \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij}k_ik_j \neq 0 \quad \Rightarrow \quad \text{stability}$$

$$A \neq 0 \quad \Rightarrow \quad \text{instability}$$

The stability criteria for a fourth-order resonance relation are

$$|A| < \left| \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij}k_ik_j \right| \Big/ \sqrt{k_0^{k_0} \cdots k_n^{k_n}} \quad \Rightarrow \quad \text{stability}$$

$$|A| > \left| \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij}k_ik_j \right| \Big/ \sqrt{k_0^{k_0} \cdots k_n^{k_n}} \quad \Rightarrow \quad \text{instability}$$

The stability of a system on the borderline of any of the given criteria is determined by fifth or higher degree terms in the expansion of its Hamiltonian.

It is important to mention that the stability results presented here have been developed strictly for Hamiltonian systems. In practice, however, flexible structures are actually non-Hamiltonian because of the presence of damping. It has been shown[2,5] that for the majority of cases damping leads to unstable motion for spin about the minor axis. Nevertheless, the results presented here are of practical value to the investigation of stability for lengths of time where the energy dissipated by damping is negligible. Such is the case for the problem that originally motivated this work.

Example Problem

The spin stability of the flexible structure shown in Fig. 2 is investigated in this section for rotation about the minor axis. The structure consists of a particle of mass m connected to a carrier body C of mass M_c by a spring with spring constant k. The unit vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ are parallel to the central principal axes of C. The position $\boldsymbol{p}$ of the particle relative to the mass center of C is given by

$$\boldsymbol{p} = (l - lx_1)\boldsymbol{b}_1 \tag{73}$$

where the constant l is a given characteristic length, e.g., the dimension $\hat{l}$ if $\hat{l} \neq 0$. It is assumed that $x_1 = 0$ when the spring is undeformed. The central principal moments of inertia of C are denoted by I_1, I_2, and I_3.

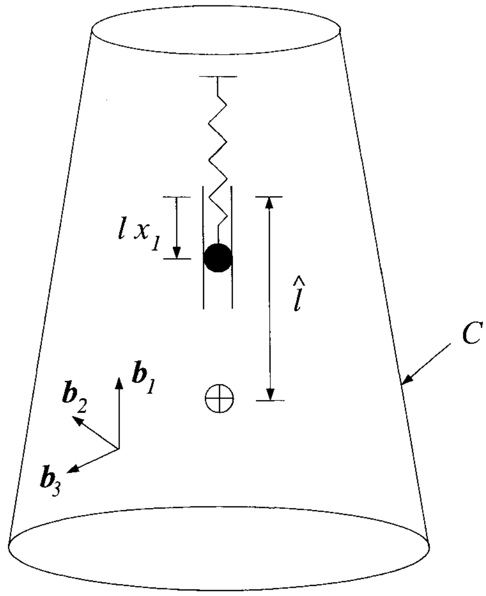

Fig. 2 Model of the flexible structure used in the example.

The kinetic energy and strain energy of the structure are given by

$$T = \tfrac{1}{2}\{I_1\omega_1^2 + [I_2 + \hat{m}l^2(\beta - x_1)^2]\omega_2^2 + [I_3 + \hat{m}l^2(\beta - x_1)^2]\omega_3^2 + \hat{m}l^2\dot{x}_1^2\} \tag{74}$$

and

$$U = \tfrac{1}{2}kl^2x_1^2 \tag{75}$$

where

$$\alpha = \frac{m}{M_c + m} \tag{76}$$

$$\beta = \hat{l}/l \tag{77}$$

$$\hat{m} = m(1 - \alpha) \tag{78}$$

Application of the procedure for transforming the equations of motion to an equivalent Hamiltonian system yields

$$G \overset{(46,74\text{–}75)}{=} \frac{1}{2}\left\{1 + \left(\frac{I_1/\bar{I}_2}{1 + (\hat{m}l^2/\bar{I}_2)f(x_1)} - 1\right)m_2^2 + \frac{I_1}{\hat{m}l^2}y_1^2 + \left(\frac{I_1/\bar{I}_3}{1 + (\hat{m}l^2/\bar{I}_3)f(x_1)} - 1\right)m_3^2 + \frac{kl^2}{I_1\Omega^2}x_1^2\right\} \tag{79}$$

$$H \overset{(30,40\text{–}41)}{=} \frac{1}{2}\left\{1 + \left(\frac{I_1/\bar{I}_2}{1 + (\hat{m}l^2/\bar{I}_2)f(x_1)} - 1\right)y_0^2\left(1 - \frac{x_0^2 + y_0^2}{4}\right) + \frac{I_1}{\hat{m}l^2}y_1^2 + \left(\frac{I_1/\bar{I}_3}{1 + (\hat{m}l^2/\bar{I}_3)f(x_1)} - 1\right)x_0^2\left(1 - \frac{x_0^2 + y_0^2}{4}\right) + \frac{kl^2}{I_1\Omega^2}x_1^2\right\} \tag{80}$$

where

$$\bar{I}_k = I_k + \hat{m}l^2\beta^2 \qquad (k = 2, 3) \tag{81}$$

$$f(x_1) = -2\beta x_1 + x_1^2 \tag{82}$$

As an aside, we note that the Hamiltonian given in Eq. (80) is integrable for $\bar{I}_2 = \bar{I}_3$. The quadratic, cubic, and quartic terms in the expansion of the Hamiltonian are

$$H_2(x, y) = \frac{1}{2}\left[\left(\frac{I_1}{\bar{I}_3} - 1\right)x_0^2 + \left(\frac{I_1}{\bar{I}_2} - 1\right)y_0^2 + \frac{kl^2}{I_1\Omega^2}x_1^2 + \frac{I_1}{\hat{m}l^2}y_1^2\right] \tag{83}$$

$$H_3(x, y) = \beta\frac{\hat{m}l^2}{I_1}\left[\left(\frac{I_1}{\bar{I}_3}\right)^2 x_0^2 x_1 + \left(\frac{I_1}{\bar{I}_2}\right)^2 y_0^2 x_1\right] \tag{84}$$

$$H_4(x, y) = \frac{1}{8}\left(1 - \frac{I_1}{\bar{I}_3}\right)x_0^4 + \frac{1}{8}\left(1 - \frac{I_1}{\bar{I}_2}\right)y_0^4 x_1 + \frac{1}{8}\left(2 - \frac{I_1}{\bar{I}_2} - \frac{I_1}{\bar{I}_3}\right)x_0^2 y_0^2 + \frac{1}{2}\frac{\hat{m}l^2}{I_1}\left(\frac{I_1}{\bar{I}_3}\right)^2\left[4\beta^2\frac{\hat{m}l^2}{I_1}\left(\frac{I_1}{\bar{I}_3}\right) - 1\right]x_0^2 x_1^2 + \frac{1}{2}\frac{\hat{m}l^2}{I_1}\left(\frac{I_1}{\bar{I}_2}\right)^2\left[4\beta^2\frac{\hat{m}l^2}{I_1}\left(\frac{I_1}{\bar{I}_2}\right) - 1\right]y_0^2 x_1^2 \tag{85}$$

The dimensionless quantities $I_1/\bar{I}_2$ and $I_1/\bar{I}_3$ are both less than unity for spin about the minor axis. Accordingly, the characteristic frequencies determined from normalization of $H_2(x, y)$ are given by

$$\Omega_0 \overset{(83)}{=} -\sqrt{(1 - I_1/\bar{I}_2)(1 - I_1/\bar{I}_3)} \tag{86}$$

$$\Omega_1 \overset{(83)}{=} \sqrt{\frac{kl^2}{I_1\Omega^2}\frac{I_1}{\hat{m}l^2}} \tag{87}$$

It turns out that of all possible order resonance relations, the only two that can lead to instability in this example are

$$2\Omega_0 + \Omega_1 = 0$$

$$\Omega_0 + \Omega_1 = 0$$

This conclusion follows from the application of the stability criteria developed in the previous section.

Table 1 Physical parameters and initial conditions used in Figs. 3–5 and 7. Initial conditions for m_1 are given by $m(0) = [1 - m_3^2(0) - m_3^2(0)]^{1/2}$.

Figure(s)	I_2/I_1	I_2/I_1	I_2/I_1	I_2/I_1	β	$m_2(0)$	$m_3(0)$	$x_1(0)$	$y_1(0)$
3,4	2.0	1.3	0.15	0.0996	1	0.035	0	0	0.010
5a	2.0	1.3	0.15	0.1046	1	0.035	0	0	0.010
5b	2.0	1.3	0.15	0.1046	1	0.071	0	0	0.020
7a	2.0	1.5	0.20	0.0449	1	0.045	0	0	0.015
7b	2.0	1.1	0.20	0.0252	1	0.045	0	0	0.015

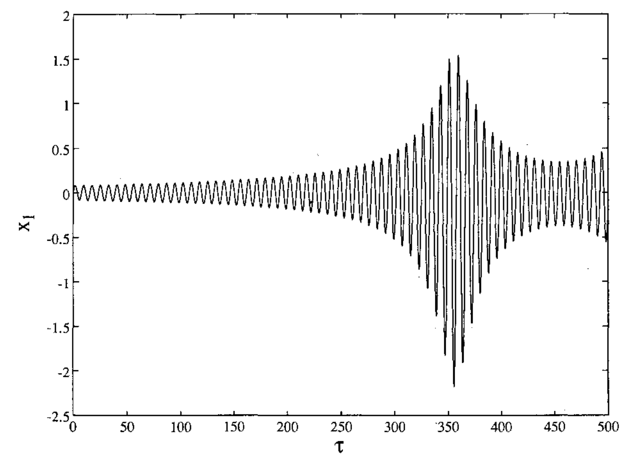

Fig. 3 Variation of x_1 for the resonance relation $2\Omega_0 + \Omega_1 = 0$.

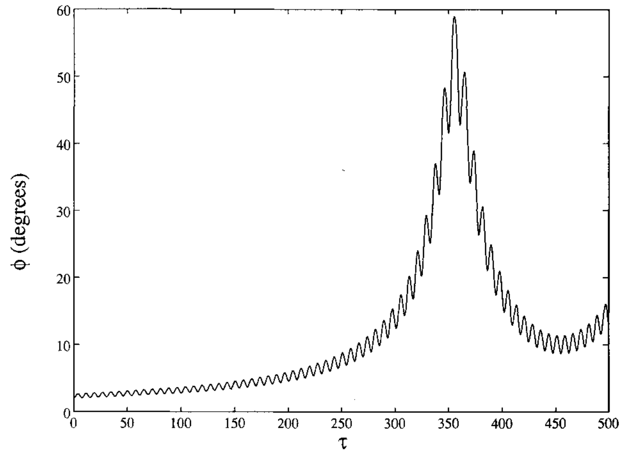

Fig. 4 Variation of ϕ for the resonance relation $2\Omega_0 + \Omega_1 = 0$.

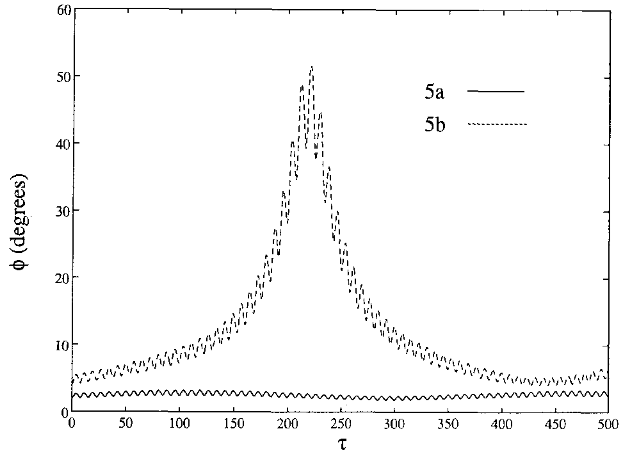

Fig. 5 Variation of ϕ for the resonance relation $2.05\Omega_0 + \Omega_1 = 0$.

Case I: $2\Omega_0 + \Omega_1 = 0$

Normalization of the Hamiltonian through degree three shows that the constant A appearing in Eq. (72) is given by

$$A = \frac{\beta[(\hat{m}l^2/I_1)(kl^2/I_1\Omega^2)]^{1/4}}{2\sqrt{2}}\left[\frac{(I_1/\bar{I}_3)^2}{1 - I_1/\bar{I}_3} - \frac{(I_1/\bar{I}_2)^2}{1 - I_1/\bar{I}_2}\right] \tag{88}$$

According to the stability criteria for third-order resonance relations, the simple spin is stable only if $A = 0$. This condition corresponds physically to either an axisymmetric body ($I_2 = I_3$) or the rest position of the particle coinciding with the carrier body mass center ($\beta = 0$).

Simulation results are presented in Figs. 3–5 to help illustrate the behavior of the structure when the resonance relation $2\Omega_0 + \Omega_1 = 0$ is satisfied or nearly satisfied. Shown in the figures are plots of the generalized coordinate x_1 and the angle ϕ [see Eq. (47)] as functions of the dimensionless time variable τ. Results were obtained from numerical integration of Eqs. (3–5) and (8) using the parameter values and initial conditions reported in Table 1. Initial conditions were chosen so that the energy and momentum correspond to those for a simple spin.

It is evident from Figs. 3 and 4 that satisfaction of the resonance relation leads to instability. Notice that although the motion is unstable, the magnitude of x_1 is bounded. This result follows from the fact that the energy stored in the spring can never exceed the total energy of the system.

Plots of the angle ϕ are shown in Fig. 5 for $2.05\Omega_0 + \Omega_1 = 0$. The simulations associated with the two plots in the figure are identical in every respect except for the choice of the initial conditions (see Table 1).

Lyapunov stability is expected since no fourth or lower order resonance relations are satisfied. The stable behavior displayed in Fig. 5a is consistent with this expectation. In contrast, the large growth of the angle ϕ in Fig. 5b is somewhat unexpected. The simple spin is indeed Lyapunov stable, but for practical purposes the motion may be considered unstable.

The observations of the previous paragraph show that a low-order resonance relation need not be satisfied exactly for unstable motion to occur. Such considerations are important in practical applications, especially when the values of the characteristic frequencies are only known approximately.

Case II: $\Omega_0 + \Omega_1 = 0$

Normalization of the Hamiltonian through degree four results in the following expressions for the nonzero constants appearing in Eq. (66).

$$a_{00} = \frac{1}{4}\left(2 - \frac{I_1}{\bar{I}_2} - \frac{I_1}{\bar{I}_3}\right) - \beta^2 \frac{\hat{m}l^2}{I_1}\left[\frac{5(b_2^2 + b_3^2)}{12} + \frac{7b_2b_3}{6}\right] \tag{89}$$

$$a_{01} + a_{10} = \frac{-(b_2 + b_3)}{2} + \beta^2 \frac{\hat{m}l^2}{I_1}\left[b_2\left(\frac{2I_1}{\bar{I}_2} + \frac{2b_2}{3}\right) + b_3\left(\frac{2I_1}{\bar{I}_3} + \frac{2b_3}{3}\right) - \frac{4b_2b_3}{3}\right] \tag{90}$$

$$A = \frac{(b_2 - b_3)}{4} + \beta^2 \frac{\hat{m}l^2}{I_1}\left[b_3\left(\frac{I_1}{\bar{I}_3} + b_3\right) - b_2\left(\frac{I_1}{\bar{I}_2} + b_2\right)\right] \tag{91}$$

where

$$b_k = \frac{(I_1/\bar{I}_k)^2}{(1 - I_1/\bar{I}_k)} \qquad (k = 2,3) \tag{92}$$

According to the stability criteria for the $\Omega_0 + \Omega_1 = 0$ resonance relation, spin about the minor axis is stable if

$$|a_{00} + a_{01} + a_{10}| > |A| \tag{93}$$

Stability diagrams in the $I_1/\bar{I}_2$, $I_1/\bar{I}_3$ parameter space are shown in Fig. 6 for $\beta^2(\hat{m}l^2/I_1) = 0.2$. Boundaries of stability are obtained by finding the locus of points that satisfies the equation

$$|a_{00} + a_{01} + a_{10}| = |A| \tag{94}$$

Regions labeled as S and U are stable and unstable, respectively. Points within regions labeled as N are not physically realizable; the sum of two principal moments of inertia in these regions does not exceed the third.

Results from two different simulations are presented in Fig. 7 as a check of the stability diagram. Specific values of the parameters $I_1/\bar{I}_2$ and $I_1/\bar{I}_3$ used in the simulations correspond to the two marked points in Fig. 6. Values of all the parameters and initial conditions are reported in Table 1. The stable and unstable behaviors exhibited in the figure are consistent with the stability diagram.

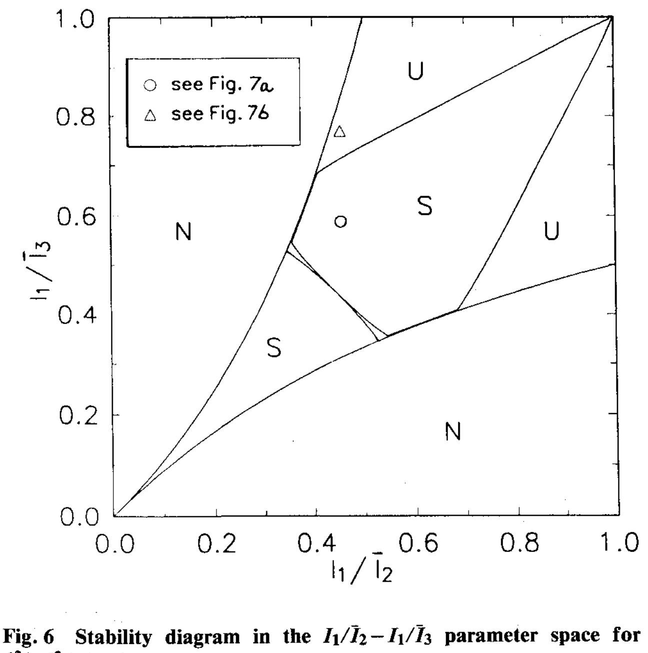

Fig. 6 Stability diagram in the $I_1/\bar{I}_2 - I_1/\bar{I}_3$ parameter space for $\beta^2(\hat{m}l^2/I_1) = 0.2$.

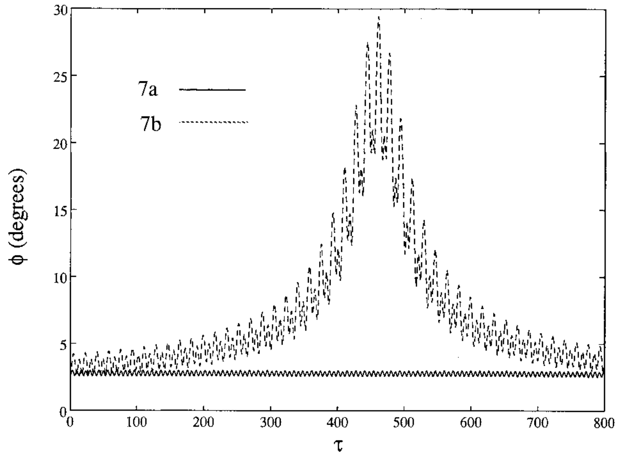

Fig. 7 Variation of for the resonance relation $\Omega_0 + \Omega_1 = 0$.

Conclusions

The nonlinear stability of undamped flexible structures free of applied forces and moments is investigated for spin about the minor axis. A procedure is developed which allows the equations of motion to be expressed as a Hamiltonian system. Using this equivalence, the existing theory for Hamiltonian systems is applied to the issue of spin stability.

It is shown that a motion of simple spin which is stable in the linear approximation may be unstable when nonlinear terms are included. For linearly stable systems, the existence of at least one low-order resonance relation among the characteristic frequencies is typically required for instability. Stability criteria are developed for systems satisfying a single resonance relation of order four or less. These criteria are applied to the stability analysis of an example problem and confirmed by numerical integration of the equations of motion.

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